Application of Optimal Control to Perfect Model Following

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Linear optimal control is applied to a model-following problem where a plant, subject to parameter variations, is required to follow the response of a given model to an arbitrary input. A fixed-gain control configuration is obtained for any model input that provides "perfect model following" for a given set of plant parameters. The sensitivity of the system to plant parameter variations is reduced by formulating a type-one perfect model-following system. The existence of feedforward gains for perfect model following is discussed and the results are applied to an aircraft control problem.

Introduction

In some control systems applications it is desirable that plant, containing uncertain parameters or parameters that vary with operating conditions, dynamically respond like a given model in the entire range of operating conditions. This objective is met through model-following systems.

Two types of model-following systems are formulated in Tyler[1] for a regular problem using optimal control theory. In the first system, the dynamics of the plant, $\dot{x} = Fx + Gu$ (x-state, u-control), are matched with the dynamics of a conceptual model, $\dot{\eta} = L\eta$, by minimizing the norm $\|\dot{y} - Ly\|$ of the plant output y (y is related to the state x by the linear transformation $y = Hx$). This system provides a good matching between model and plant transfer functions at high frequencies.

The second system is formulated by minimizing the squared error between plant and model outputs. The system thus obtained provides uniform matching between model and plant transfer functions for all frequencies and is less sensitive to plant parameter variations. Therefore, squared error minimization will be used in the model-following formulation presented here.

When the model is excited by an arbitrary input, model-following techniques developed for the regulator case lead to an optimal-control law that depends on the model input description. This requires a different control law for every conceivable model input. In practical applications, however, it is desirable to use a fixed-gain control configuration for every possible model input. Such a system can be derived optimally from a performance index that contains model input rate constraints.

If the optimal-control law is derived from a performance index containing error and control constraints, good model following can be achieved only through high feedback gains. To eliminate the high-gain requirement for good model following, additional terms must be added to the performance index. It will be shown below that the inclusion of cost functionals $x_P{}^TSx_m$, $x_P{}^TLu_m$ to the performance index, where the weighting matrices S and L depend on the Riccati matrices, leads to a perfect model-following system at any feedback gain. Of course, such a system can be derived only for a given set of plant parameters.

The perfect model-following system is similar to the cancellation compensation concept used in classical control theory which can be obtained directly from the state equations. However, as indicated in Kalman[2] it is of interest to know the performance index from which a given control-system configuration derives optimally. Therefore, optimal-control theory will be used in the formulation of a perfect model-following system.

Model Following with Arbitrary Model Input

The dynamics of the time-invariant plant and model are governed by the following linear state equations:

$$\text{plant } \dot{x}_P = F_Px_P + G_Pu \tag{1a}$$

$$\text{model } \dot{x}_m = F_mx_m + G_mu_m \tag{1b}$$

where x_P is the n-dimensional output of a controllable plant, x_m is the n-dimensional output of a given model, u_m is the m-dimensional model input and u is the r-dimensional linearly independent control.

To meet the control objectives, the following quadratic performance index is formulated:

$$V = \tfrac{1}{2}\int_0^\infty [x^T\phi x + u^TRu + 2x^T\psi\dot{x}]dt$$

where

$$x = \begin{bmatrix} x_P \\ x_m \\ u_m \end{bmatrix}; \quad \phi = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{12}{}^T & \phi_{22} & 0 \\ \phi_{13}{}^T & 0 & 0 \end{bmatrix}; \quad \psi = \begin{bmatrix} 0 & 0 & \psi_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2}$$

and where R is a given positive-definite symmetric weighting matrix, x is the augmented state vector and ϕ_{ij}'s are constant weighting matrices to be determined later.

The optimal control u that minimizes the performance index in (2) is obtained from Pontryagin's maximum principle.[3]

The Hamiltonian H is defined here as

$$H = \tfrac{1}{2}[x^T\phi x + u^TRu] + \lambda^T(F_Px_P + G_Pu) + x^T\psi\dot{x} \tag{3}$$

where λ is the n-dimensional costate vector that enforces the constraint imposed by the plant dynamics in Eq. (1a). Terms associated with model dynamics and the dynamical description of the model input do not contribute to the optimal-control law, hence, omitted in the definition of the Hamiltonian in Eq. (3). Substituting the optimal-control law $u = -R^{-1}G^T\lambda$ into Eq. (3) and differentiating H with respect to x_P and λ yields the canonical equations

$$\begin{aligned} \dot{x}_P &= F_Px_P - G_PR^{-1}G_P{}^T\lambda \\ \dot{\lambda} + \phi_{11}x_P + \phi_{12}x_m + \phi_{13}u_m + F_P{}^T\lambda + \psi_{13}\dot{u}_m &= 0 \end{aligned} \tag{4}$$

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Substituting $\lambda = P_{11}x_P + P_{12}x_m + P_{13}u_m$ in Eq. (4) where P_{11}, P_{12}, and P_{13} are constant Riccati matrices and requiring that the canonical equations be identically satisfied for every x_P, x_m, u_m, and $\dot{u}_m$ yields the following Riccati equations:

$$P_{11}F_P + F_P{}^TP_{11} + \phi_{11} - P_{11}\Lambda P_{11} = 0 \tag{5a}$$

$$(F_P - \Lambda P_{11})^TP_{12} + P_{12}F_m = -\phi_{12} \tag{5b}$$

$$(F_P - \Lambda P_{11})^TP_{13} = -\phi_{13} - P_{12}G_m \tag{5c}$$

$$\psi_{13} = -P_{13} \tag{5d}$$

where $\Lambda = G_PR^{-1}G_P{}^T$. The term $2x_P{}^T\psi_{13}\dot{u}_m$ (or $2x^T\psi\dot{x}$) is included in the performance index in Eq. (2) in order to eliminate the dependence of the optimal-control law on the model input description. The resulting optimal-control system contains fixed gains for any possible model input.

The optimal-control law is

$$u = -R^{-1}G_P{}^T(P_{11}x_P + P_{12}x_m + P_{13}u_m) \tag{6}$$

The existence of a unique set of Riccati matrices P_{11}, P_{12}, and P_{13} that yield a stable model-following system is discussed in Theorem 1.

Theorem 1

a) A unique feedback gain matrix, $K_P = R^{-1}G_P{}^TP_{11}$, exists that minimizes the performance index in (2). P_{11} is positive-definite symmetric and the closed-loop plant is stable if and only if 1) the plant (F_P,G_P) is completely controllable and 2) R is positive definite and ϕ_{11} is nonnegative definite.

b) A unique set of feedforward gain matrices, $K_m = -R^{-1}G_P{}^TP_{12}$ and $K_v = -R^{-1}G_P{}^TP_{13}$ exist that minimize the performance index in (2), if and only if

$$\lambda_i + \mu_j \neq 0, \quad i,j = 1,2, \ldots$$

where λ_i is an eigenvalue of the closed-loop plant $(F_P - G_PR^{-1}G_P{}^TP_{11})$ and μ_j is an eigenvalue of the model F_m.

Proof

The proof of Eq. (5a) of the theorem is given in Kalman.[4] Equation (5b) is of the form $P_{12}A + BP_{12} = Q$, hence, part b of the theorem follows directly from the proof given in Bellman.[5]

Type-Zero Perfect Model-Following System

The quality of model following usually depends on the feedback gains as demonstrated by the following example:

Consider the second order model-following system

$$\text{plant} \quad \begin{bmatrix}\dot{x}_{P1}\\ \dot{x}_{P2}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -b_P & -a_P\end{bmatrix}\begin{bmatrix}x_{P1}\\ x_{P2}\end{bmatrix} + \begin{bmatrix}0\\ b_P\end{bmatrix}u$$
$$\text{model} \quad \begin{bmatrix}\dot{x}_{m1}\\ \dot{x}_{m2}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -b_m & -a_m\end{bmatrix}\begin{bmatrix}x_{m1}\\ x_{m2}\end{bmatrix} + \begin{bmatrix}0\\ b_m\end{bmatrix}u_m \tag{7}$$

where u_m is a step input ($\dot{u}_m = 0$) and define the performance index

$$V = \frac{1}{2}\int_0^\infty [q(x_{P1} - x_{m1})^2 + q(x_{P2} - x_{m2})^2 + u^2]dt \tag{8}$$

This performance index clearly reflects model following since the error $(x_P - x_m)$ and the control u are constrained. The response of this system to a step model input is plotted in Fig. 1 for various q's. Observation of this figure shows that better model following can only be achieved through high gains.

To eliminate the high-gain requirement for good model-following, one must include additional terms in the performance index. Consider the index

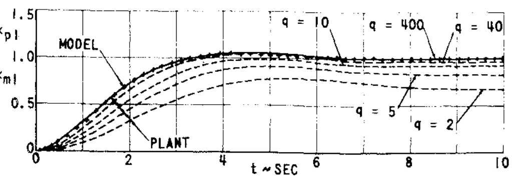

Fig. 1 Step response of a second-order system.

$$V = \frac{1}{2}\int_0^\infty [x^T\phi x + u^TRu - 2x_P{}^TP_{13}\dot{u}_m]dt$$

where

$$x = \begin{bmatrix}x_P\\ x_m\\ u_m\end{bmatrix}; \quad \phi = \begin{bmatrix}N^TQ_1N & -N^TQ_1N + S & L\\ -N^TQ_1N + S^T & N^TQ_1N & 0\\ L^T & 0 & 0\end{bmatrix} \tag{9}$$

and where N is a linear transformation that defines a new set of output vectors $y_P = Nx_P$, $y_m = Nx_m$, Q_1 is a positive-definite symmetric weighting matrix and S and L are weighting matrices that provide perfect model following as shown below. The form of the performance index in (9) reflects model following between the output vectors y_P and y_m.

On substituting ϕ from Eq. (9) into (5), one gets the Riccati equations

$$P_{11}F_P + F_P{}^TP_{11} + N^TQ_1N - P_{11}\Lambda P_{11} = 0 \tag{10a}$$

$$P_{12}F_m + F_P{}^TP_{12} - N^TQ_1N + S - P_{11}\Lambda P_{12} = 0 \tag{10b}$$

$$P_{12}G_m + F_P{}^TP_{13} + L - P_{11}\Lambda P_{13} = 0 \tag{10c}$$

where $\Lambda = G_PR^{-1}G_P{}^T$. Selecting S and L as a function of the Riccati matrices P_{ij}

$$S = -(P_{11} + P_{12})F_m - F_P{}^T(P_{11} + P_{12}) \tag{11a}$$

$$L = -(P_{11} + P_{12})G_m - F_P{}^TP_{13} \tag{11b}$$

reduces Eqs. (10b) and (10c) to

$$\begin{aligned} G_{PO}R^{-1}G_{PO}{}^TP_{13} &= -G_m \\ G_{PO}R^{-1}G_{PO}{}^T(P_{11} + P_{12}) &= F_{PO} - F_m \end{aligned} \tag{12}$$

Now, substituting the optimal-control u from Eq. (6) into the plant state Eq. (1a) and using Eq. (12), one gets

$$(d/dt)(x_P - x_m) = -(G_{PO}R^{-1}G_{PO}{}^TP_{11} - F_{PO})(x_P - x_m)$$

The solution of this homogeneous differential equation with initial conditions, $x_P(0) = x_m(0)$ is $x_P(t) = x_m(t)$, $0 \leq t \leq \infty$, which implies perfect model following at any feedback gain.

The "perfect model-following system" is derived from a quadratic performance index that contains cross-product terms of plant output states and model input and output states. The weighting matrices associated with these cross-product terms, S and L, are related to the Riccati matrices through Eq. (11).

The perfect model-following system could have been obtained directly from the state equations without using optimal control. However, it is of interest to know the performance index from which such a system is derived optimally as indicated previously. Figure 2 shows the block diagram of a perfect model-following system.

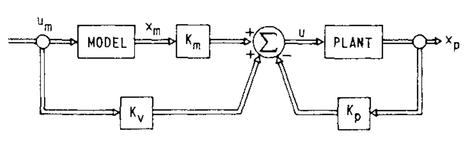

Fig. 2 Block diagram type-zero perfect model-following system.

The feedback gain, $K_P = R^{-1}G_P{}^TP_{11}$, obtained from Eq. (10a), has a unique positive-definite symmetric solution P_{11} by Theorem 1a. The existence of feedforward gains K_m and K_v for perfect model following are discussed in Theorem 2 below.

Theorem 2

When the model and plant are of the same order, a unique set of feedforward gains exist for perfect model following.

a) If G_P is invertible, the plant follows any model perfectly.

b) If the number of linearly independent controls is equal to or larger than the number of linearly independent model inputs ($r \geq m$), the plant follows a transformed model $T_PT_m{}^{-1}x_m(t)$, or those models that are related to the plant by Eq. (17).

Proof

If G_P is invertible, P_{12}, P_{13} and the feedforward gains K_v, K_m, can be obtained directly from Eq. (12).

$$\begin{aligned} -R^{-1}G_P{}^TP_{13} &= K_v = G_P{}^{-1}G_m \\ -R^{-1}G_P{}^TP_{12} &= K_m = K_P + G_P{}^{-1}(F_m - F_P) \end{aligned} \tag{13}$$

If G_P is a nonsquare matrix of rank r, there exist transformation matrices T_P and T_m that transform the plant and model to phase-variable canonical forms[6,7] given by

$$F_P{}^0 = \overbrace{\begin{bmatrix} \hat{F}_0 \\ F_P \end{bmatrix}}^{n} \begin{matrix} \}\, n-r \\ \}\, r \end{matrix} \qquad G_P{}^0 = \overbrace{\begin{bmatrix} 0 \\ \bar{G}_P \end{bmatrix}}^{r} \begin{matrix} \}\, n-r \\ \}\, r \end{matrix} \tag{14a}$$

$$F_m{}^0 = \overbrace{\begin{bmatrix} F_0 \\ F_0 \\ \hat{F}_m \end{bmatrix}}^{n} \begin{matrix} \Big\}\, n-m \\ \\ \}\, m \end{matrix} \qquad G_m{}^0 = \overbrace{\begin{bmatrix} 0 \\ \hat{G}_m \end{bmatrix}}^{m} \begin{matrix} \}\, n-m \\ \}\, m \end{matrix} \tag{14b}$$

where $\bar{G}_P$ and $\hat{G}_m$ are invertible and F_0, $\hat{F}_0$ are of the form

$$\begin{bmatrix} \hat{F}_0 \\ F_0 \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 1 & & 0 \\ \vdots & & \vdots & & \ddots & \\ 0 & \cdots & 0 & 0 & & 1 \end{bmatrix}$$

If the number of linearly independent controls is equal to or larger than the number of linearly independent model inputs, $r \geq m$, the transformed model $F_m{}^0$ can be rearranged by shifting rows as follows:

$$\begin{aligned} F_m{}^0 &= \overbrace{\begin{bmatrix} F_0 \\ \bar{F}_m = \begin{bmatrix} \hat{F}_0 \\ \hat{F}_m \end{bmatrix} \end{bmatrix}}^{n} \begin{matrix} \}\, n-r \\ \Big\}\, r \end{matrix} \\ G_m{}^0 &= \overbrace{\begin{bmatrix} 0 \\ \bar{G}_m = \begin{bmatrix} 0 \\ \hat{G}_m \end{bmatrix} \end{bmatrix}}^{m} \begin{matrix} \}\, n-r \\ \Big\}\, r \end{matrix} \end{aligned} \tag{15}$$

Now, substituting (14a) and (15) into (12) yields the desired feedforward gains

$$\begin{aligned} K_v &= \bar{G}_P{}^{-1}G_m \\ K_m &= K_P + \bar{G}_P{}^{-1}(\bar{F}_m - \bar{F}_P) \end{aligned} \tag{16}$$

In this case, P_{12}, P_{13} are not unique, but, the feedforward gains K_v, K_m are unique and the plant follows perfectly the transformed model, $x_P(t) = T_PT_m{}^{-1}x_m(t)$. In order for the plant to follow the model perfectly, it is necessary for the submatrices of G_P, F_P, etc., to satisfy the following relationship:

$$\begin{aligned} G_{P1}\bar{G}_P{}^{-1}\bar{G}_m &= G_{m1} \\ G_{P1}\bar{G}_P{}^{-1}(\bar{F}_P - \bar{F}_m) &= F_{P1} - F_{m1} \end{aligned} \tag{17}$$

where G_{P1}, $\bar{G}_P$, etc., are defined by the partitioning of G_P, . . ., etc., as

$$G_P = \overbrace{\begin{bmatrix} G_{P1} \\ \bar{G}_P \end{bmatrix}}^{r} \begin{matrix} \}\, n-r \\ \}\, r \end{matrix} \qquad F_P = \overbrace{\begin{bmatrix} F_{P1} \\ \bar{F}_P \end{bmatrix}}^{n} \begin{matrix} \}\, n-r \\ \}\, r \end{matrix}$$

$$G_m = \overbrace{\begin{bmatrix} G_{m1} \\ \bar{G}_m \end{bmatrix}}^{m} \begin{matrix} \}\, n-r \\ \}\, r \end{matrix} \qquad F_m = \overbrace{\begin{bmatrix} F_{m1} \\ \bar{F}_m \end{bmatrix}}^{n} \begin{matrix} \}\, n-r \\ \}\, r \end{matrix}$$

If the plant and model are in phase-variable canonical form[6] $G_{P1} = 0$, the conditions in Eq. (17) reduce to $G_{m1} = 0$ and $F_{m1} = F_{P1}$. This concludes the proof of the Theorem.

The equations of motion of most vehicles can be represented in phase-variable canonical form where $(n - r)$ state variables are the derivatives of the other state variables. If the plant has as many linearly independent controls as degrees of freedom (G_P has full rank) and if the equations of motion of the model are similar to those of the plant, perfect model following is always possible.

Optimal-control theory yields what is commonly known as a feedforward-feedback model-following system. Other model-following systems such as inverted-model and response feedback, derived from classical methods, require proper selection of feedback gains for good model following. With a perfect model-following system one has complete freedom in selecting the feedback gains to satisfy other design criteria, such as to reduce sensitivity to plant parameter variations. Therefore, this system is superior to other model-following systems cited previously.

The step response of the second-order system described by Eqs. (7) and (8) is shown in Fig. 3 where one observes that the plant response matches exactly the model response for all feedback gains.

Of course, perfect model-following is possible only for one set of plant parameters, F_{P0} and G_{P0}, referred to as nominal, for which the system gains are computed. When F_P, G_P vary from the nominal for different operating conditions, high feedback gains still are required to reduce the error, $x_P - x_m$. However, the purpose of high gains here is not to provide good model following, but to reduce sensitivity. A method for reducing sensitivity to plant parameter variations is discussed below.

Type-One Perfect Model-Following System

A reduction in sensitivity to plant parameter variations, over and above that obtainable with a type-zero system, can be achieved through a type-one perfect model-following system. This system insures zero steady-state error to a step model input in the entire range of operating conditions (for any F_P, G_P), in addition to providing perfect dynamic model following at the nominal operating condition (for F_{P0}, G_{P0}). A type-one model-following system is derived from a performance index that contains cost functionals of $\int x_P$, $\int x_m$ and $\int u_m$, in addition to the cost functionals included in Eq. (9).

$$V = \tfrac{1}{2}\int_0^\infty [z^T\phi z + 2z_P{}^T\psi_{13}\dot{z}_u + u^TRu]dt$$

where

$$z = \begin{bmatrix} z_P \\ z_m \\ z_u \end{bmatrix}; \quad z_P = \begin{bmatrix} \int_0^t x_P dt \\ x_P \end{bmatrix};$$
$$z_m = \begin{bmatrix} \int_0^t x_m dt \\ x_m \end{bmatrix}; \quad z_u = \begin{bmatrix} \int_0^t u_m dt \\ u_m \end{bmatrix} \tag{18}$$

The enlarged plant z_p and the enlarged model z_m evolve according to:

$$\dot{z}_P = A_P z_P + B_P u \tag{19a}$$

$$\dot{z}_m = A_m z_m + B_m u_m \tag{19b}$$

where

$$A_P = \begin{bmatrix} 0 & I \\ 0 & F_P \end{bmatrix}; \quad A_m = \begin{bmatrix} 0 & I \\ 0 & F_m \end{bmatrix};$$
$$B_P = \begin{bmatrix} 0 \\ G_P \end{bmatrix}; \quad B_m = \begin{bmatrix} 0 & 0 \\ 0 & G_m \end{bmatrix} \tag{19c}$$

Equations (18) and (19) are of the same form as those in Eqs. (1) and (2) for the type-zero model-following system, hence, letting

$$\lambda = P_P z_P + P_m z_m + P_u z_u$$
$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{12}^T & \phi_{22} & 0 \\ \phi_{13}^T & 0 & 0 \end{bmatrix} \tag{20}$$

yields a set of Riccati equations similar to those in Eq. (5)

$$P_P A_P + A_P^T P_P - P_P \varkappa P_P + \phi_{11} = 0 \tag{21a}$$

$$P_m A_m + A_P^T P_m - P_P \varkappa P_m + \phi_{12} = 0 \tag{21b}$$

$$P_m B_m + A_P^T P_u - P_P \varkappa P_u + \phi_{13} = 0 \tag{21c}$$

$$P_u + \psi_{13} = 0 \tag{21d}$$

where

$$\varkappa = B_P R^{-1} B_P^T = \begin{bmatrix} 0 & 0 \\ 0 & \Lambda \end{bmatrix}$$

The controllability matrix associated with the enlarged system is

$$\Delta = [B_P, A_P B_P, \ldots A_P^{2n-1} B_P]$$
$$= \begin{bmatrix} 0 & G_P \ldots F_P^{2n-2} G_P \\ G_P & F_P G_P \ldots F_P^{2n-1} G_P \end{bmatrix} \tag{22}$$

If G_P has rank n, the determinant of the first $2n$ columns of Δ, evaluated by Gauss's algorithm,[8] is $\det(\Delta) = \det(G_P) \cdot \det(G_P)$. The rank of the controllability matrix Δ is $2n$ and the plant is completely controllable. If G_P is not square, some state variables in the phase-variable canonical form are the derivatives of the other state variables (e.g., $\dot{x}_1 = x_2$). These state variables appear twice in the enlarged state vector z, defined in (18). Because of this redundancy, the controllability matrix has rank $<2n$, (at least two rows of Δ are identical), the enlarged plant defined in Eq. (19a) is uncontrollable and the Riccati Equation (21a) does not have a unique solution P_P by Theorem 1.

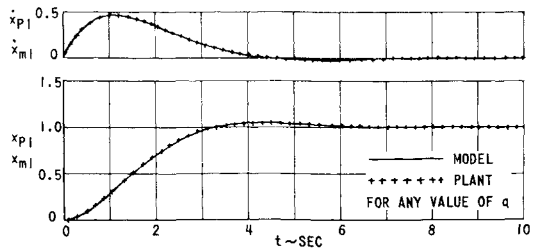

Fig. 3 Step response of a perfect model system.

To avoid this difficulty, one must define an enlarged plant, $\dot{x}_P = F_P x_P + G_P u$, and an enlarged model, $\dot{x}_m = F_m x_m + G_m u_m$ where x_P and x_m consist of the original state variables and their time integrals, with each state appearing once in the enlarged state vector. The system thus defined is a controllable type-zero system, and the Riccati Equation (5a) or (10a) have a unique solution P_{11}. The feedback gain matrix is given by $K_P = R^{-1} G_P^T P_{11}$ and feedforward gains K_m and K_v are obtained from Eq. (16) for perfect model following.

Despite the difficulty arising from the definition of the enlarged system in Eq. (19), the formulation in Eqs. (18–21) will be pursued further to get a better understanding of the properties of a type-one model-following system. Define the weighting matrix ϕ and partition the Riccati matrices P_P, P_m, and P_u as

$$P_P = \begin{array}{c} \overbrace{\quad}^{n} \overbrace{\quad}^{n} \\ \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \end{array} \begin{array}{l} \} n \\ \} n \end{array} \qquad P_m = \begin{array}{c} \overbrace{\quad}^{n} \overbrace{\quad}^{n} \\ \begin{bmatrix} P_{13} & P_{14} \\ P_{14}^T & P_{24} \end{bmatrix} \end{array} \begin{array}{l} \} n \\ \} n \end{array}$$

$$P_u = \begin{array}{c} \overbrace{\quad}^{n} \overbrace{\quad}^{n} \\ \begin{bmatrix} P_{15} & P_{16} \\ P_{25} & P_{26} \end{bmatrix} \end{array} \begin{array}{l} \} n \\ \} n \end{array}$$

$$\phi = \begin{bmatrix} N^T Q_2 N & 0 & -N^T Q_2 N & 0 & 0 & 0 \\ 0 & N^T Q_1 N & 0 & S - N^T Q_1 N & 0 & L \\ -N^T Q_2 N & 0 & N^T Q_2 N & 0 & 0 & 0 \\ 0 & S^T - N^T Q_1 N & 0 & N^T Q_1 N & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L^T & 0 & 0 & 0 & 0 \end{bmatrix} \tag{23}$$

where Q_1 and Q_2 are positive-definite weighting matrices, N is defined in Eq. (9) and S and L will be determined below. Substitute Eq. (23) into (21) and into the optimal-control law $u = -R^{-1} B_P^T (P_P z_P + P_m z_m + P_u z_u)$ to get

$$u = -R^{-1} G_P^T (P_{12}^T \textstyle\int x_P + P_{22} x_P + P_{14}^T \int x_m + P_{24} x_m + P_{25} \int u_m + P_{26} u_m) \tag{24a}$$

$$-P_{12} \Lambda P_{12}^T + N^T Q_2 N = 0 \tag{24b}$$

$$-P_{12} \Lambda P_{14}^T - N^T Q_2 N = 0 \tag{24c}$$

$$P_{12}^T + P_{22} F_P + F_P^T P_{22} + P_{12} - P_{22} \Lambda P_{22} + N^T Q_1 N = 0 \tag{24d}$$

$$P_{14}^T + P_{24} F_m + F_P^T P_{24} + P_{14} - P_{22} \Lambda P_{24} - N^T Q_1 N + S = 0 \tag{24e}$$

$$-P_{12} \Lambda P_{25} = 0 \tag{24f}$$

$$P_{24} G_m + F_P^T P_{26} + P_{16} - P_{22} \Lambda P_{26} + L = 0 \tag{24g}$$

Note from Eqs. (24b) and (24c) that $P_{14}^T = -P_{12}^T \neq 0$, implying zero steady-state error to a step model input,

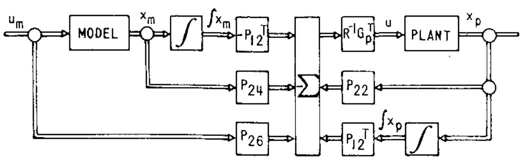

Fig. 4 Block diagram of a type-one model-following system.

($\dot x_P = \dot x_m = x_P = x_m = u_m = u = 0$) in the entire range of operating conditions (for any F_P and G_P). Further note from Eq. (24f) that $P_{25} = 0$, hence, the optimal-control law does not explicitly depend on $\int u_m$ (depends on u_m).

Now, select S and L in terms of the Riccati matrices as

$$\begin{aligned} S &= -(P_{22} + P_{24})F_m - F_P{}^T(P_{22} + P_{24}) \\ L &= -F_P{}^TP_{26} - P_{16} - (P_{22} + P_{24})G_m \end{aligned} \tag{25}$$

with this selection Eqs. (24e and f) reduce to

$$\begin{aligned} G_{PO}R^{-1}G_{PO}{}^TP_{26} &= -G_m \\ G_{PO}R^{-1}G_{PO}{}^T(P_{22} + P_{24}) &= F_{PO} - F_m \end{aligned} \tag{26}$$

On substituting (26) and (24a) into the plant dynamical equation (1a), one gets

$$(d/dt)(x_P - x_m) = (F_{P0} - \Lambda_0P_{22})(x_P - x_m) - \Lambda_0P_{12}{}^T\int(x_P - x_m)dt \tag{27}$$

where

$$\Lambda_0 = G_{PO}R^{-1}G_{PO}{}^T$$

The solution of this homogeneous differential equation with initial conditions $x_P(0) = x_m(0)$ is

$$\int_0^t x_P(\tau)d\tau = \int_0^t x_m(\tau)d\tau$$

or, $x_P(t) = x_m(t)$, at all times, implying perfect model following at the nominal operating condition for which P_{24} and P_{26} satisfy Eq. (26). Thus, one proves that the type-one system provides perfect model following at the nominal operating condition and insures zero steady-state error to a step model input in the entire range of operating conditions.

Since Eq. (26) is similar to Eq. (12), the conditions for the existence of feedforward gains, given by theorem 2, hold for the type-one model-following system. The block diagram of a type-one model-following system is shown in Fig. 4. This concludes the theoretical treatment of the problem. The results are applied below to a fourth-order aircraft control problem.

Application to an Aircraft Control Problem

For small perturbations the linearized three-degrees-of-freedom longitudinal state equations of the aircraft are given[9] by

Plant Aircraft

$$\begin{bmatrix}\dot\theta_P\\ \ddot\theta_P\\ \dot V_P\\ \dot\alpha_P\end{bmatrix} = \begin{bmatrix}0&1&0&0\\ x&x&x&M_\alpha\\ x&x&x&x\\ x&x&x&Z_\alpha\end{bmatrix}\begin{bmatrix}\theta_P\\ \dot\theta_P\\ V_P\\ \alpha_P\end{bmatrix} + \left[\bar G_P = \begin{bmatrix}0&0&0\\ \begin{bmatrix}x&x&x\\ x&D_{\delta_r}&x\\ x&x&Z_{\delta_z}\end{bmatrix}\end{bmatrix}\right]\begin{bmatrix}\delta_e\\ \delta_x\\ \delta_z\end{bmatrix} \tag{28}$$

$$\begin{bmatrix}\dot\theta_m\\ \ddot\theta_m\\ \dot V_m\\ \dot\alpha_m\end{bmatrix} = \begin{bmatrix}0\quad 1\quad 0\quad 0\\ \begin{bmatrix}\bar F_m\\ (3\times 4)\end{bmatrix}\end{bmatrix}\begin{bmatrix}\theta_m\\ \dot\theta_m\\ V_m\\ \alpha_m\end{bmatrix} + \begin{bmatrix}0\qquad 0\\ \begin{bmatrix}\bar G_m\\ (3\times 2)\end{bmatrix}\end{bmatrix}\begin{bmatrix}\delta_{e_m}\\ \delta_{T_m}\end{bmatrix}$$

Model Aircraft

where θ,V, etc., are incremental motion variables about a trim condition, θ is the pitch attitude, V is airspeed, α is angle of attack of the wing, δ_e is elevator control deflection, δ_x is throttle control deflection and δ_z is flap control deflection. The inputs to the model aircraft are elevator command δ_{e_m} and throttle command δ_{T_m}; x in Eq. (28) denotes nonzero entries.

The state equations in (28) are in phase-variable canonical form, and satisfy the perfect model-following conditions of Theorem 2. M_α, Z_α, D_{δ_r}, and Z_{δ_z} in Eq. (28) are the predominant plant parameters affecting sensitivity. The gains are computed for a given set of parameters M_α, Z_α, etc.

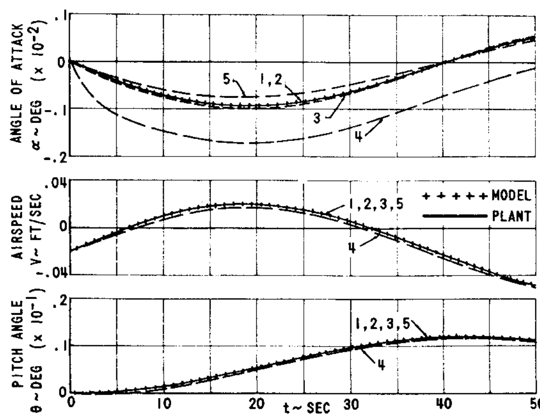

Fig. 6 Aircraft response to a unit throttle step; type-zero perfect model-following system.

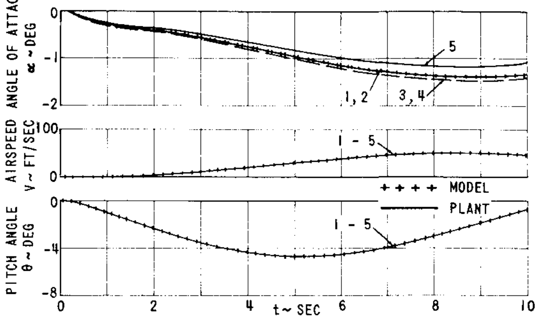

Fig. 5 Aircraft response to a unit elevator step; type-zero perfect model-following system.

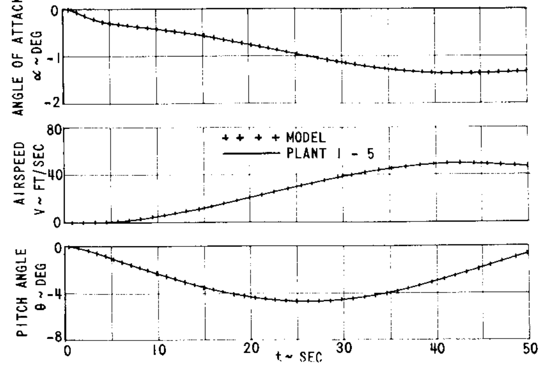

Fig. 7 Aircraft response to a unit elevator step; type-one perfect model-following system.

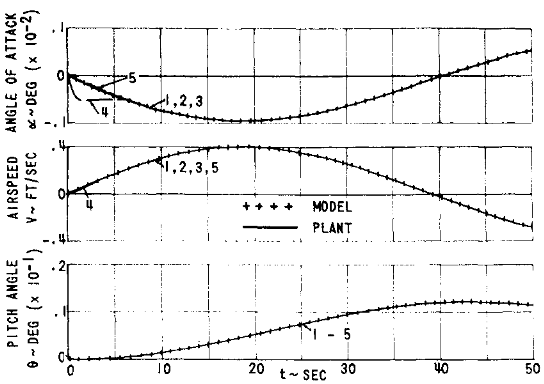

Fig. 8 Aircraft response to a unit throttle step; type-one perfect model-following system.

(nominal operation condition); these parameters are then changed to examine the sensitivity of the model-following system. The results are plotted in Figs. 5–8. Curves 1 through 5 in these figures show the aircraft response as follows: curve 1 at the nominal operating condition, curve 2 with 50% reduction in M_α, curve 3 with 10% reduction in Z_α, curve 4 with 50% reduction in D_{δ_r}, curve 5 with 50% reduction in Z_{δ_z}.

Figures 5 and 6 show the aircraft response to a unit elevator and to a unit throttle step input for a type-zero system. Perfect model following is noted from curve 1 in both figures. When parameter changes listed above are made, the plant aircraft no longer follows the model aircraft perfectly as observed from curves 3, 4, and 5 in Figs. 5 and 6; the angle of attack deviation is quite large in Fig. 6.

The aircraft response for a type-one system is shown in Figs. 7 and 8. The deviation between plant and model outputs caused by parameter changes is unnoticeably small in Fig. 7. An initial deviation between angle-of-attack time histories is noted in Fig. 8. Comparison of time histories in Figs. 5, 6, and Figs. 7, 8 shows the sensitivity improvement obtained by a type-one system.

The reader unfamiliar with aircraft dynamics may wonder why the step responses in Figs. 5–8 do not reach steady state. The open-loop dynamics of an aircraft contain poles close to the $j\omega$ axis that produce low-frequency oscillations, (phugoid dynamics) when no corrective action is applied by the pilot. The purpose of model following is not to stabilize the aircraft but to make the plant aircraft follow the model aircraft, which also contains row damping.

Conclusions

The results are summarized as follows: 1) The existence of a unique set of gains for a model-following system is established in Theorem 1. 2) A fixed-gain control configuration is obtained for any model input. A type-zero cancellation compensation system is derived by using optimal control theory. The resulting system provides perfect model following at a nominal operating condition. At any other operating condition the type-zero system requires high feedback gains for acceptable model-following performance. 3) A type-one system is formulated. This system provides perfect model following for one set of plant parameters and insures zero steady-state error to a step model input in the entire range of operating conditions. 4) The existence of feedforward gains for perfect model following is established in Theorem 2.

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